Section Agenda
Stochastic Lanchester Models

Agenda

- State spaces and time (discrete and continuous models)
- A big Continuous Time Markov Chain
- Exact solutions…are hard
- Easy to simulate
- Some pictures
- When it matters

State Space and Time

State-Space

TIME		Discrete	Continuous
	Discrete	Markov Chain	Difference Equations
	Continuous	***Today (real world)***	Lanchester Diff. Eqns.

The Idea/Defining Some Variables

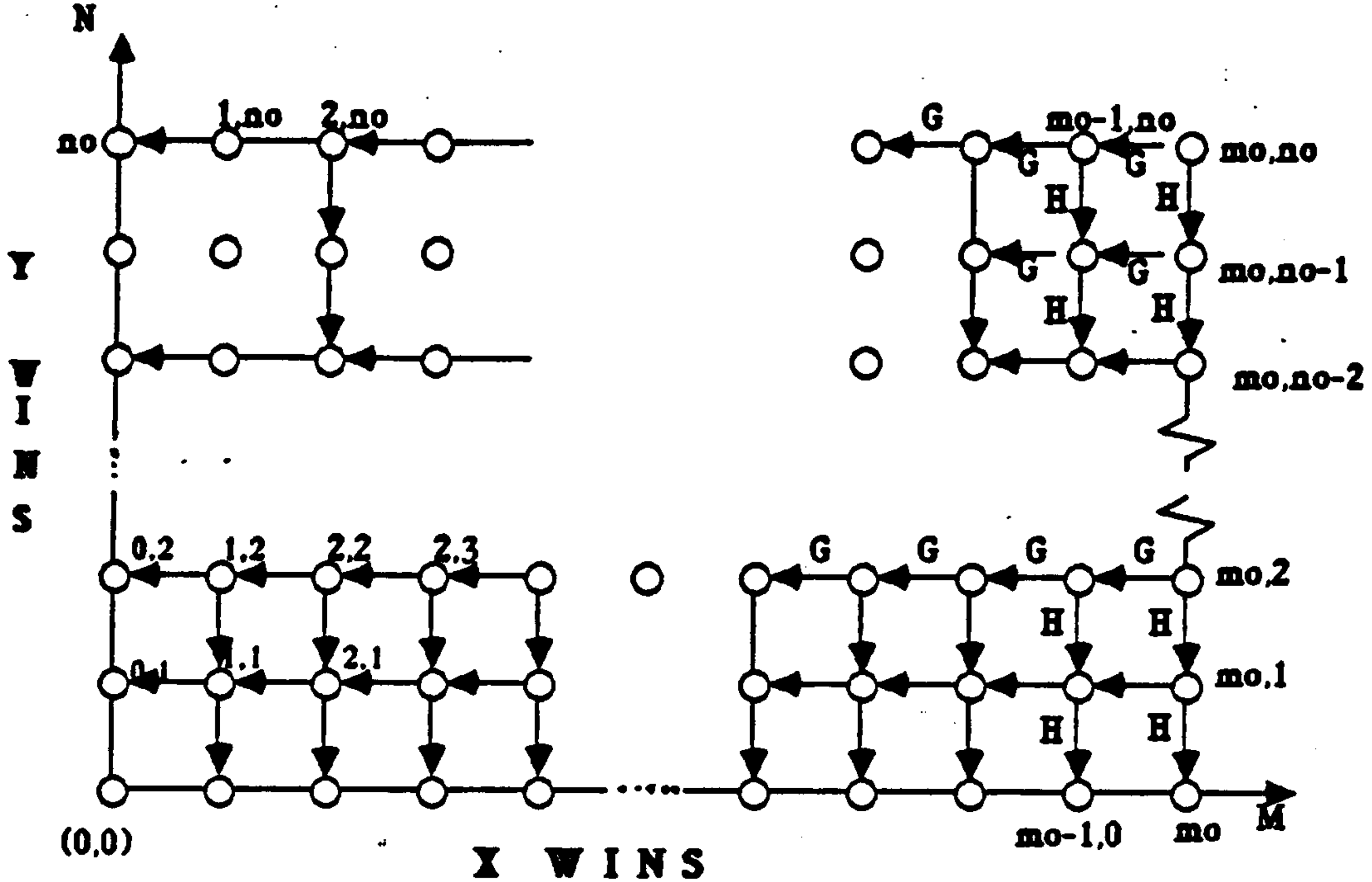
- Idea: Each side has an integer number of forces
  - $M(t) = m$ = # of X force units at time t, with $M(0) = m_0$.
  - $N(t) = n$ = # of Y force units at time t, with $N(0) = n_0$.
- Only one or no units can be killed in a small time step Δt.
- Markov assumption
- Time in state (till transition) has an exponential distribution
  - When transition and who gets killed are random

The Idea/Defining Some Variables (continued)

- Rate at which X is attrited (think: dX/dt) = $G(m,n)$
- Rate at which Y is attrited (think: dY/dt) = $H(m,n)$
- This depends on n and m, not how or when we got there
- Assume Poisson Processes for casualties
  - $P(1 \text{ X casualty in } \Delta t) = G(m,n) \times \Delta t$
  - $P(1 \text{ Y casualty in } \Delta t) = H(m,n) \times \Delta t$
  - $P(2 \text{ or more casualties in } \Delta t) = 0$
- This implies time till next casualty in exponential

- See picture

Possible States and Transitions



Some Facts About Exponential RVs

- Let T_1 = time till the next X casualty
  - $T_1 \sim \text{Exp}(\lambda_1)$
- Let T_2 = time till the next Y casualty
  - $T_2 \sim \text{Exp}(\lambda_2)$
- Assume T_1 and T_2 are independent
- Then,
  - Time till next casualty $\sim \text{Exp}(\lambda_1 + \lambda_2)$
  - P(next casualty is X) = $\lambda_1 / (\lambda_1 + \lambda_2)$
- Let's prove these

More Assumptions/Definitions

- This is a death process, of course
  - Can solve explicitly by Kolmogorov's equations (Very hard…for me at least)
  - Very easy to simulate (lots of reps required)
- Let $P(m,n,t) = P(M=m \text{ and } N=n \text{ at time } t)$
- $P(m,n,t+\Delta t) = P(m,n,t) \times P(\text{no casualties in time } \Delta t) + P(m+1,n,t) \times P(1 \text{ X casualty in time } \Delta t) + P(m,n+1,t) \times P(1 \text{ Y casualty in time } \Delta t)$
- See Picture

From Conditional Prob and Our Assumptions (I)

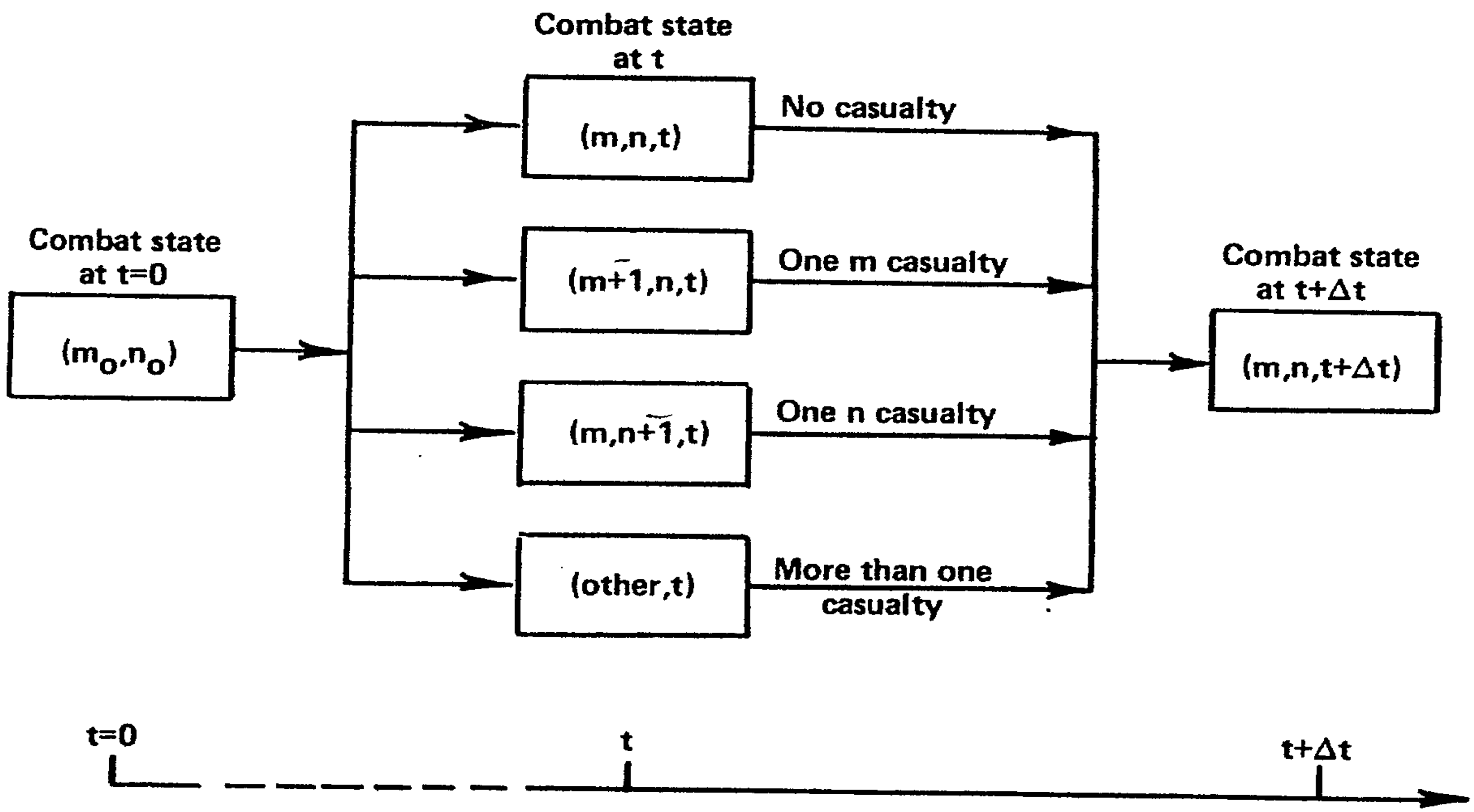


Fig. 5.6 Possible combat states.

For Lanchester Situations

- Area Fire (mean rate of kill)
  - $G(m,n) = amn$
  - $H(m,n) = bmn$
- Aim Fire (mean rate of kill)
  - $G(m,n) = an$
  - $H(m,n) = bm$
- Add your favorite rate here:___________

To Simulate Stochastic Area Fire

- For area fire
  - Time till next casualty is distributed as exponential with rate = amn + bmn, note: mean time to a kill = 1/rate
  - Draw a random exponential to find time of next kill
- Pr(X killed|kill) = amn / (amn + bmn) = a/(a+b)
- Pr(Y killed|kill) = bmn / (amn + bmn) = b/(a+b)
- Repeat from new state until stopping conditions (e.g., breakpoints) are met
- What's the probability that the next 3 killed are X?
  - General statements can be solved as a binomial

To Simulate Stochastic Aimed Fire

- For aimed fire
  - Time till next casualty is distributed as exponential with rate = an + bm, note: mean time to a kill = 1/rate
  - Draw a random exponential to find time of next kill
- Pr(X killed|kill) = an / (an + bm)
- Pr(Y killed|kill) = bm / (an + bm)
- Repeat from new state until stopping conditions (e.g., breakpoints) are met

Let’s Do an Example

- Lanchester Deterministic:
  - Let $X(0) = 20$, $Y(0) = 40$, $a = .01$, and $b = .02$

- Result: Y wins with 28.28 survivors at time $t = 63.32$
  - (From our formulas earlier)

- Let’s first look at the first step and then…
- let’s look at plots at time 20, 40, and 60
  - Note: at time $t=0$ we have a point mass
  - See pictures
  - What do you see?

Simulating the First Casualty

- Time of first casualty ~ Exp(an + bm)
- Plug in: get Exp(.01*40 + .02*20) = Exp(.8)
- Draw random number to get the time…
- P(First Casualty is an X) = an/(an+bm)
  .4/(.8) = .5
- Flip coin to see who is killed
- Repeat for the next casualty (with reduced m or n)

Figure 1: Time = 20

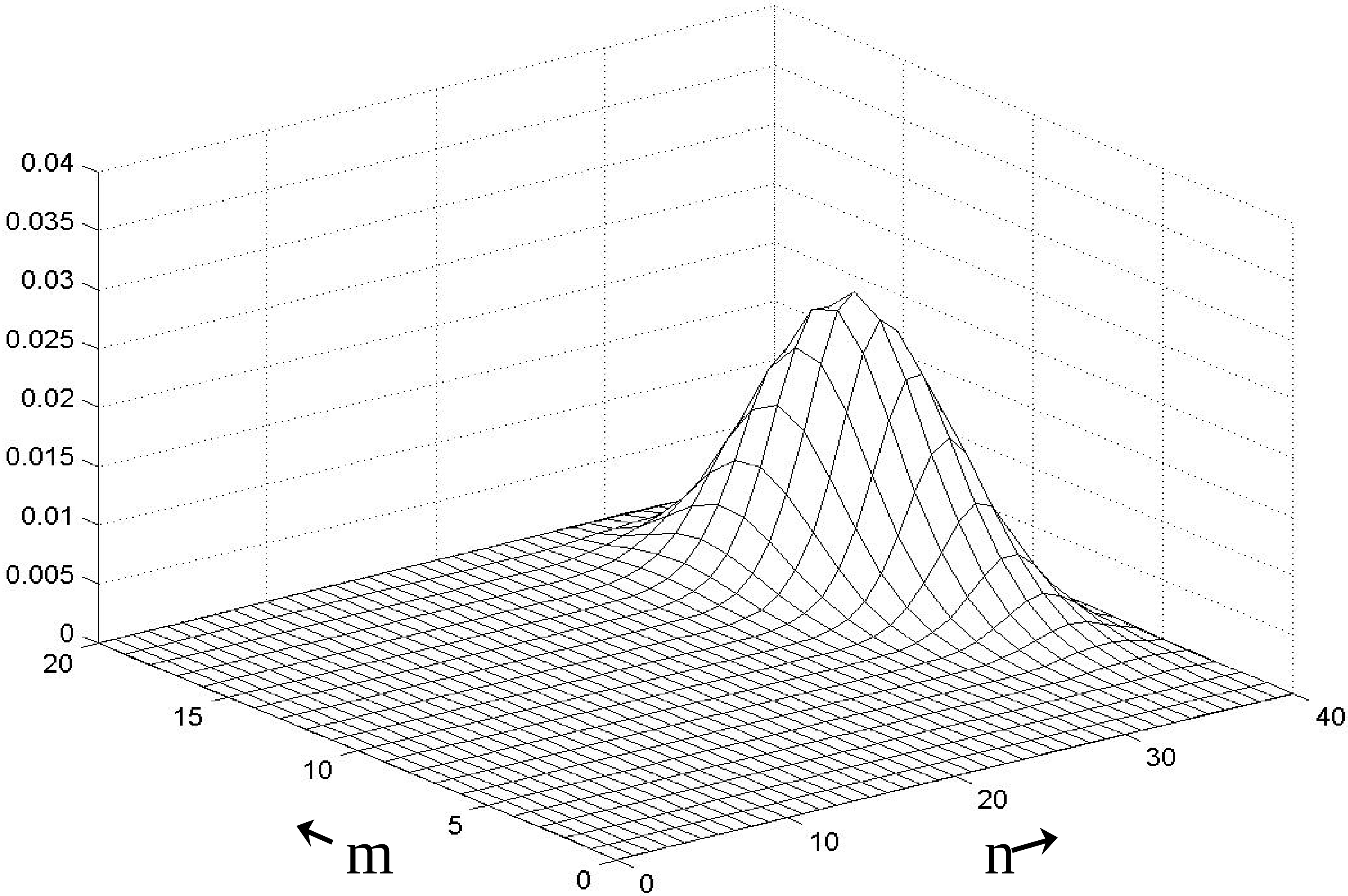


Figure 2: Time = 40

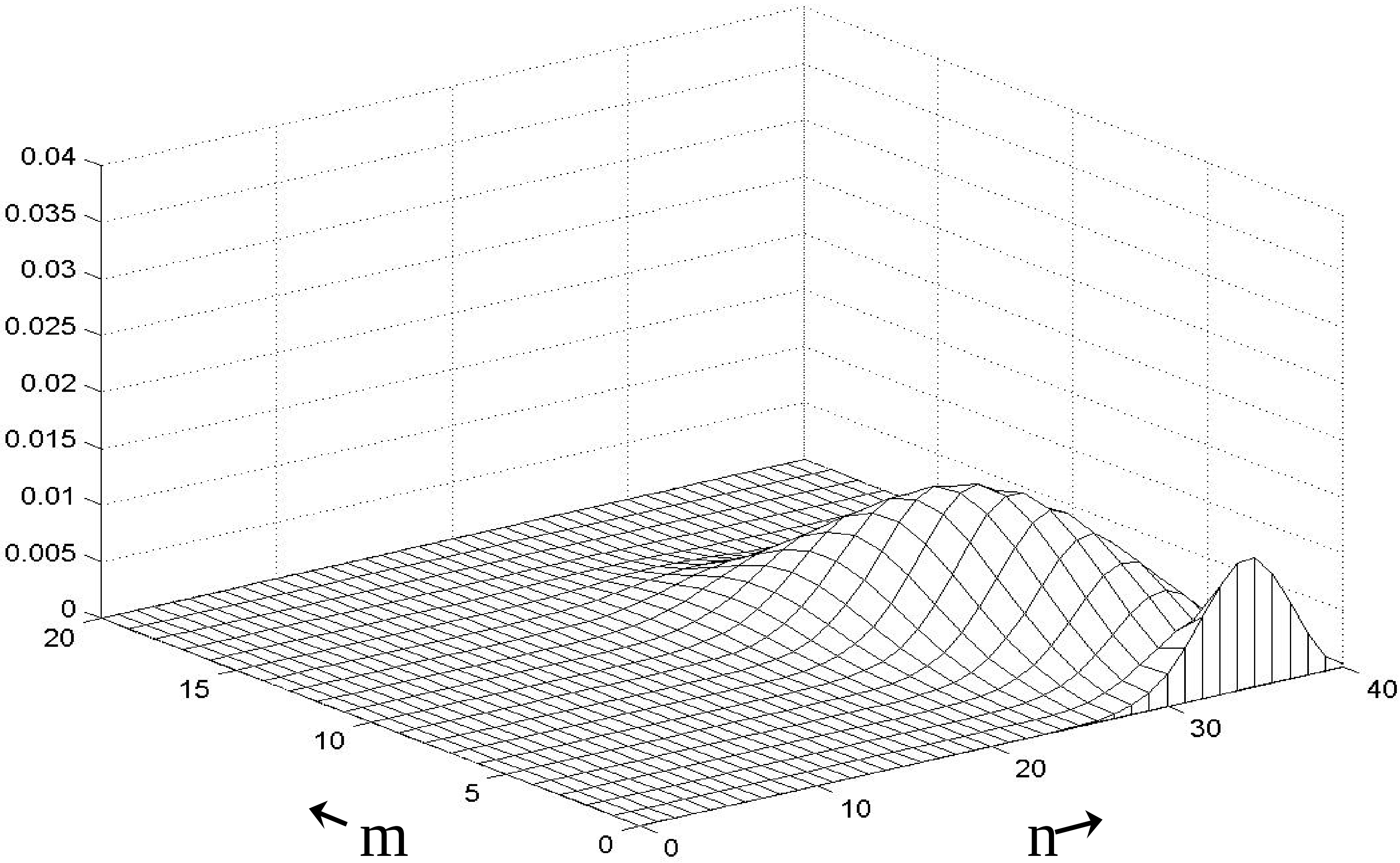
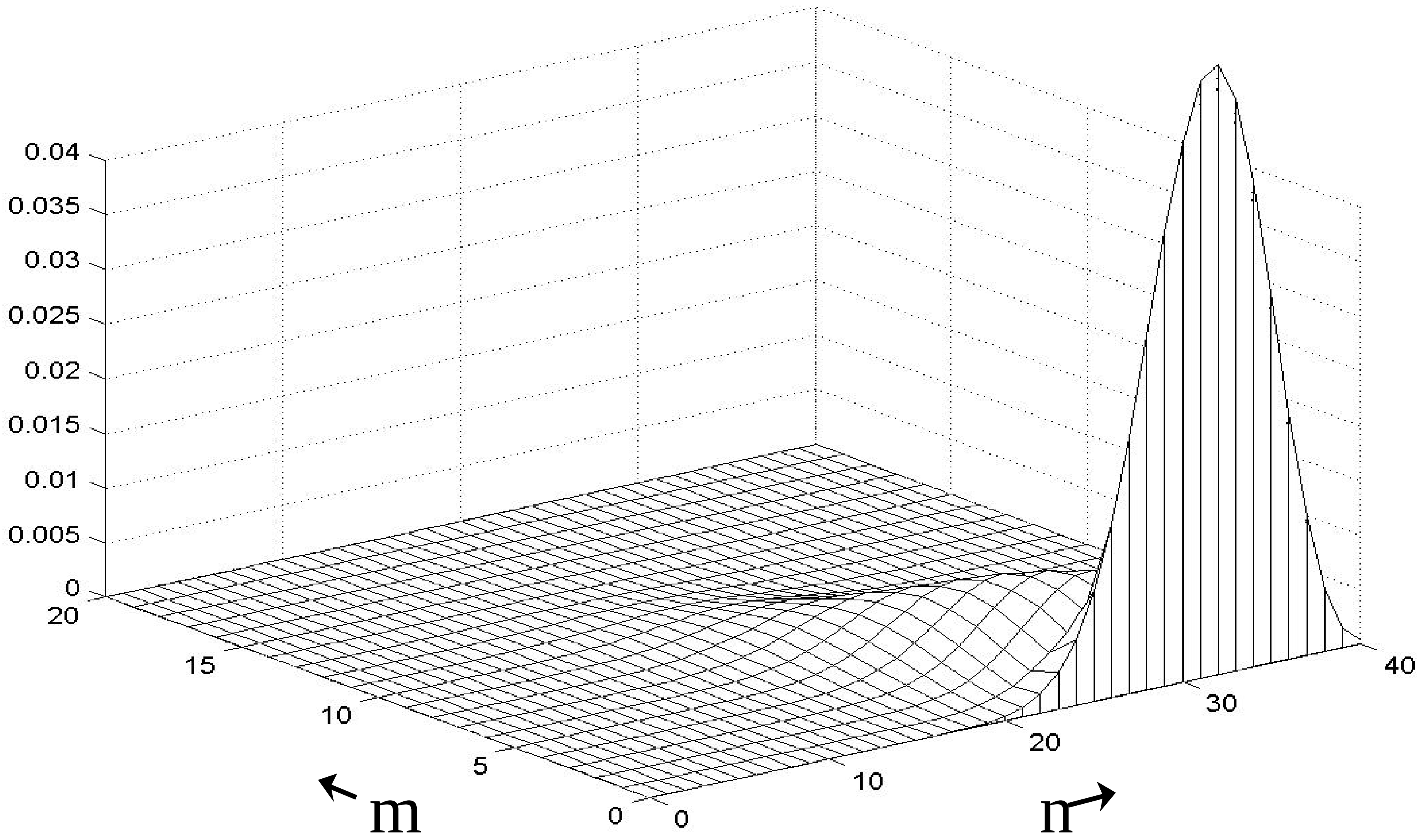


Figure 3: Time = 60



When is the Deterministic Lanchester a Good Approximation to the Stochastic Lanchester?

- Taylor (1983), supported with additional references, found that the difference between the deterministic course of combat and the stochastic mean is small for:
  - (1) highly aggregated “simple” Lanchester models, with
  - (2) sufficiently large forces (both > 20), when
  - (3) each side is willing to take substantial casualties, and
  - (4) the sides are not near parity.

I Believe in Stochastic Models

– Even if the deterministic model correctly models the “mean course of combat,” it is often more appropriate to reason probabilistically. In risk adverse situations may be more interested in the tails than the mean (See figure from Paul Davis, below)